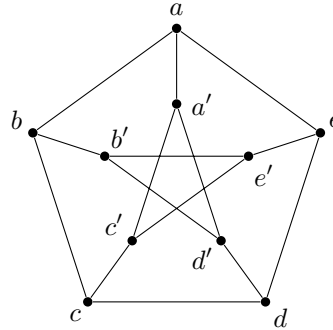


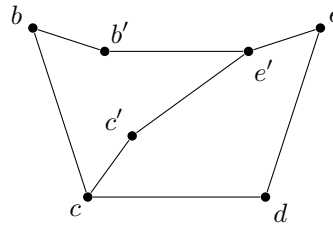
1. Is the Petersen graph a Hamiltonian graph?<sup>1</sup>

**Solution:** Consider the Petersen graph  $G$  with the labelling below:



Assume to the contrary that there is a Hamilton cycle  $C$  in  $G$ . Observe that by shifting the cycle's starting point we can assume that  $C$  starts at a vertex on the outer cycle (i.e. we start at a vertex without a “prime” on it). Now at some point in  $C$  we jump from the outer cycle to the inner cycle (i.e. there is a subsequence  $(x, x')$  in  $C$ ). Again shifting (and relabelling if necessary), we can assume  $C$  starts with  $(a, a')$ .

The next vertex in the cycle is either  $c'$  or  $d'$ , so (reflecting the graph if necessary) we may assume that  $C$  starts  $(a, a', d')$ . Now, the next vertex in the graph is either  $b'$  or  $d$ . Because  $C$  is a Hamilton cycle, in either case both vertices  $b'$  and  $d$  are the ends of a subpath  $P$  of  $C$ , so let's consider the possibilities for  $P$ ; we consider the following graph  $G'$ , obtained from  $G$  by removing the vertices we have already visited:



Now the paths from  $d$  to  $b'$  of the desired type are all given below:

$$P_1 = (d, c, b, b') \quad P_2 = (d, c, c', e', b') \quad P_3 = (d, e, e', b') \quad P_4 = (d, e, e', c', c, b, b')$$

Now  $P$  is one of the above  $P_i$  or their reverses. We finish the proof by cases.

*Case 1:* If  $P = P_1$ , then  $C$  starts out with either  $(a, a', d', d, c, b, b', e')$  or  $(a, a', d', b', b, c, d, e)$ ; the extra vertices at the end are the only available neighbors to move to next. Now in either case we must visit  $c'$ , which will trap us at  $c'$ . Hence this case is impossible.

*Case 2:* If  $P = P_2$ , then  $C$  starts out with either  $(a, a', d', d, c, c', e', b', b)$  or  $(a, a', d', b', e', c', c, d, e)$ ; the extra vertices at the end are the only available neighbors to move to next. In the former case it is impossible to visit  $e$ , and in the latter case it is impossible to visit  $b$ . Hence this case is impossible.

*Case 3:* If  $P = P_3$ , then  $C$  starts out with either  $(a, a', d', d, e, e', b', b)$  or  $(a, a', d', b', e', e, d, c)$ ; the extra vertices at the end are the only available neighbors to move to next. In either case, we must at some point visit  $c'$ , which will cut off any remaining route to  $a$ . Hence this case is impossible.

*Case 4:* If  $P = P_4$ , then  $C$  starts out with either  $(a, a', d', d, e, e', c', c, b, b')$  or  $(a, a', d', b', b, c, c', e', e, d)$ . In either case,  $C$  has must return to  $a$  next; neither  $b'$  nor  $d$  have an edge to  $a$ . Hence this case is impossible.

All cases lead to impossibility, so a Hamilton cycle  $C$  cannot possibly exist. Hence  $G$  is not Hamiltonian.

<sup>1</sup>It will probably be helpful to draw pictures of the paths we construct below to see what is happening as you go.